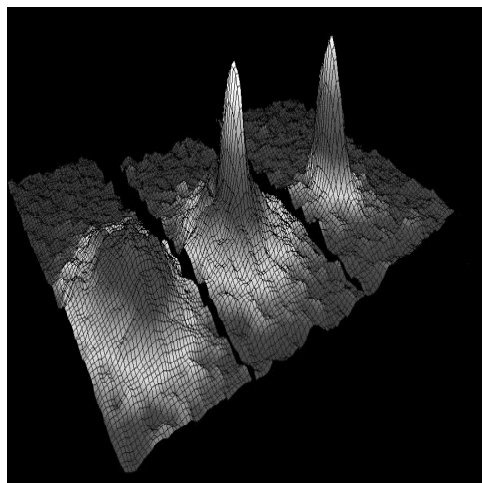


Lecture 8

Bose-Einstein Condensation

8.1 Background

- Originally predicted by Albert Einstein and Satyendra Bose in 1924, this phenomenon was only first observed in 1995 by Carl Wiemann and Eric Cornell using rubidium-87. Since 1999, a number of other atomic gases have been shown to demonstrate BE condensation, including sodium, lithium, and hydrogen.
- In the first experiment, approximately 10^4 atoms were confined in a volume of $\sim 10^{-15}$ using laser cooling and magnetic trapping.
- The laser cooling is insufficient to cool the atoms to the required temperatures, so *evaporative* cooling was subsequently performed to get the temperature down to about 10^{-7} K. At this temperature, a large number of atoms formed a condensate,



We have already discussed the generic properties associated with Bose-Einstein condensation: Namely, that since we are dealing with bosons (i.e. integral spin, symmetric wavefunctions), we can have more than one particle in a given single-particle state.

- Since all particles will be in their lowest lying single-particle states at $T = 0$, it must be that all particles are in the *same* lowest single-particle state.

- This rapid accumulation of atoms in the ground state is called **Bose-Einstein condensation**. This is different than the phenomenon of condensation that occurs at a vapour-liquid phase boundary, since there is no *spatial* separation of phases with different properties.
- The collection of atoms in this state is called the **condensate**.
- The onset of Bose-Einstein condensation actually occurs at a critical temperature, T_c , referred to as the **condensation temperature** or the **degeneracy temperature** (whereby, the BE gas at temperatures below T_c is called degenerate).

8.2 Number of Bosons

Let us return to our expression (Eq. 7.12) derived for the total number of electrons in the gas and modify this expression to account for spin-zero bosons. Hence, we must:

1. Use the BD distribution
2. Eliminate the factor of 2, which accounted for the 2 possible spin orientations of our electrons \rightarrow now there is only one spin orientation

With these changes, the total number of bosons is given as

$$\frac{N}{V} = \frac{2\pi (2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon \quad (8.1)$$

which is valid for energies (temperatures) large compared to the ground-state energy, ϵ_1 . Note that the above integral can *not* be solved analytically.

- Let us first observe that $\mu < 0$ for a BE gas. Why? It goes back to our original calculation of the grand partition function,

$$\begin{aligned} \Xi &= \prod_{i=1}^{\infty} \Xi_i \\ \Xi_i &= \sum_{n_i} e^{\beta(\mu-\epsilon_i)n_i} \end{aligned} \quad (8.2)$$

where the summation is just a geometric series.

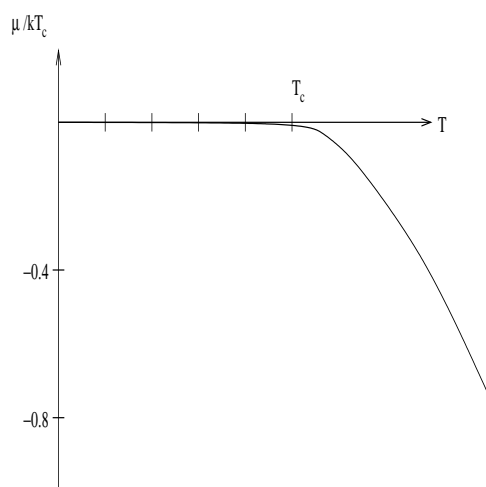
- In order for this series to converge, we require that $e^{\beta(\mu-\epsilon_i)} < 1 \rightarrow$ which requires that $\mu < \epsilon_i$.
- Since the ground-state energy, ϵ_1 , is the lowest lying energy (which we nominally set as the zero of the energy scale), then this condition will hold for *all* single-particle states if it holds true for the ground-state. Namely, $\mu < 0$.

- Alternatively, considering the BE distribution,

$$\bar{n}_{\text{BE}}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}, \quad (8.3)$$

we require $\mu < 0$ to guarantee that we don't obtain negative occupation numbers!

- Let us now *fix* the concentration N/V and vary T
- Then if the R.H.S. of the above equation is to remain constant as we decrease T , the chemical potential, which is negative, must *increase* (or becomes less negative) *ra* that is, $|\mu|$ must *decrease*.
- This would imply that at a certain temperature, T_c , $\mu = 0$. Putting this all together, we obtain the following picture,



Let us consider the integral above (Eq. 8.1) at this point,

$$\frac{N}{V} = \frac{2\pi (2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{(\epsilon/kT_c)} - 1}$$

Make the following substitution $x = \epsilon/kT_c$ and the above expression becomes,

$$\begin{aligned} \frac{N}{V} &= \left(\frac{2\pi mkT_c}{h^2} \right)^{3/2} \left\{ \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2} dx}{e^x - 1} \right\} \\ &= 2.61 \times \left(\frac{2\pi mkT_c}{h^2} \right)^{3/2} \end{aligned} \quad (8.4)$$

- But this would imply that for temperatures $T < T_c$, the total number of bosons *decreases* \rightarrow an absurdity that we have encountered before.

- To reconcile this paradox, we first remember that we converted a discrete sum over single-particle states,

$$N = \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}, \quad (8.5)$$

to an integral in deriving Eq. 8.1.

- Although the discrete sum explicitly includes the ground state, $\epsilon_1 = 0$, Eq. 8.1 neglects this state because the factor $\epsilon^{1/2}$ in the integrand gives it zero weight.
- This is not a problem at higher temperatures, but is a catastrophe for $T < T_c$, where almost all the particles are in the ground state.
- More correctly we should modify Eq. 8.1 to account for the ground-state contribution by retaining at least the first term in the sum 8.5¹,

$$N = N_1 + N_{\epsilon > 0} \quad (8.6)$$

where,

$$N_1 \equiv \frac{1}{e^{-\beta\mu} - 1} \quad (8.7)$$

corresponds to the number of particles in the ground state ($\epsilon = \epsilon_1 = 0$, and,

$$N_{\epsilon > 0} \equiv \frac{2\pi V (2m)^{3/2}}{h^3} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} \quad (8.8)$$

is the number of bosons in excited states $\epsilon > 0$.

- As mentioned above, for temperatures above T_c , the chemical potential is negative most particles will be in excited states and, hence, N_1 will be negligible. It is therefore valid to use our original equation 8.1 to determine the chemical potential.
- For $T < T_c$, the chemical potential is very close to zero and we may solve Eq. 8.8, with $\mu = 0$, to obtain,

$$N_{\epsilon > 0} = 2.62V \left(\frac{2\pi m k T}{h^2} \right)^{3/2} \quad (T < T_c) \quad (8.9)$$

We can write this expression in a more convenient form, by comparison with Eq. 8.4, to give the fraction of particles with $\epsilon > 0$,

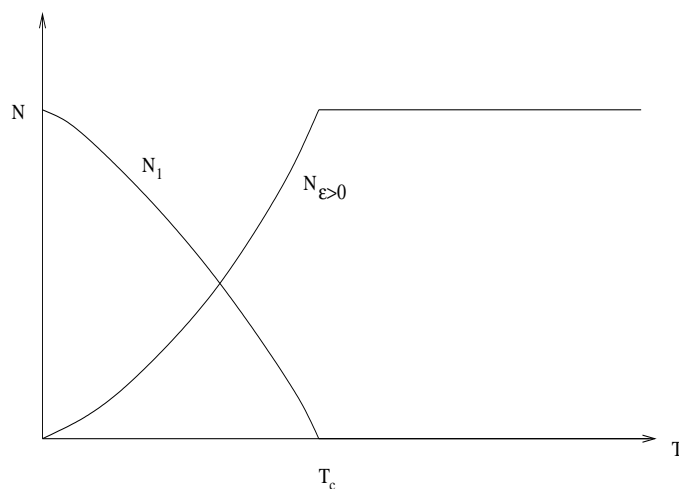
$$N_{\epsilon > 0} = N \left(\frac{T}{T_c} \right)^{3/2} \quad (T < T_c) \quad (8.10)$$

¹It has been argued that it is sufficient to keep only the first term in the sum for sufficiently large N ; however, it is not immediately clear that this is true if the spacing of the low lying energy levels is small and they are reasonably well populated.

Similarly, we can obtain the fraction of particles in the ground state as,

$$\begin{aligned} N_1 &= N - N_{\epsilon > 0} \\ &= N \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right] \quad (T < T_c) \end{aligned} \quad (8.11)$$

These boson numbers are plotted in the figure below as a function of temperature,



- The bosons in the condensate have zero *momentum*, and therefore they do not contribute to the pressure and do not have viscosity (which is related to momentum transport).
- Consider now the energy and heat capacity of our BE gas. The general expression for the energy is just,

$$U = V \left(\frac{2\pi m}{h^2} \right)^{3/2} \left\{ \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\epsilon^{3/2} d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} \right\} \quad (8.12)$$

For $T < T_c$ only the *excited* atoms ($N_{\epsilon > 0}$) have energy. Set $\mu = 0$ and make the substitution $x = \epsilon/kT$ and the above integral can be evaluated to be,

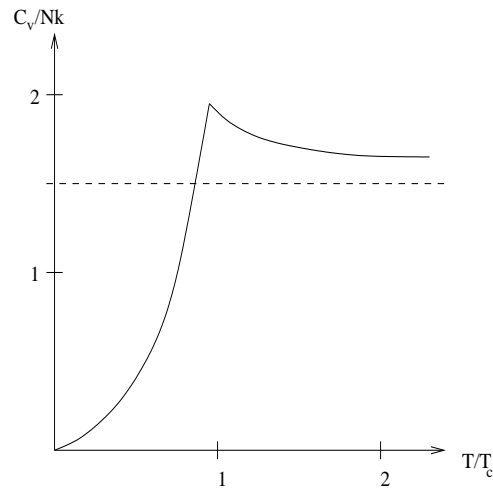
$$\begin{aligned} U &= V \left(\frac{2\pi m}{h^2} \right)^{3/2} (kT)^{5/2} \left\{ \frac{2}{\sqrt{\pi}} \cdot 1.783 \right\} \\ &= 0.77 N k \frac{T^{5/2}}{T_c^{3/2}}, \quad (T < T_c) \end{aligned} \quad (8.13)$$

where the last step follows from Eq. 8.4.

- The heat capacity may now be trivially evaluated to give,

$$C_V = 1.926 N k \left(\frac{T}{T_c} \right)^{3/2}, \quad (T < T_c), \quad (8.14)$$

which is plotted in the figure below.



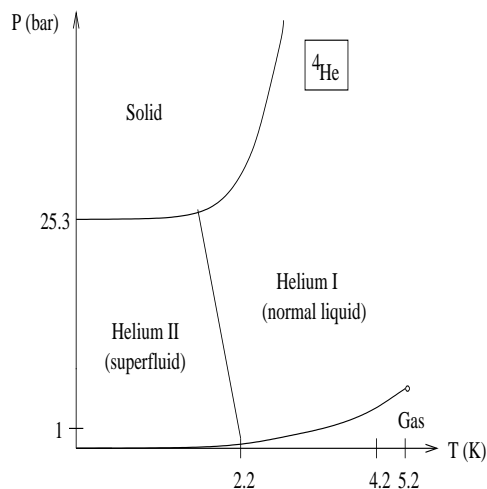
- For higher temperatures, we observe that C_V approaches its classical limiting value of $\frac{3}{2}Nk$, as predicted by the equipartition theorem.
- At T_c , the heat capacity *exceeds* its classical value.
- Note that it would require a proper treatment to show how C_V for $T < T_c$ joins with C_V for $T > T_c$. Although the curve is *continuous* at $T = T_c$, there is a kink there.

8.3 BEC for $^4\text{Helium}$

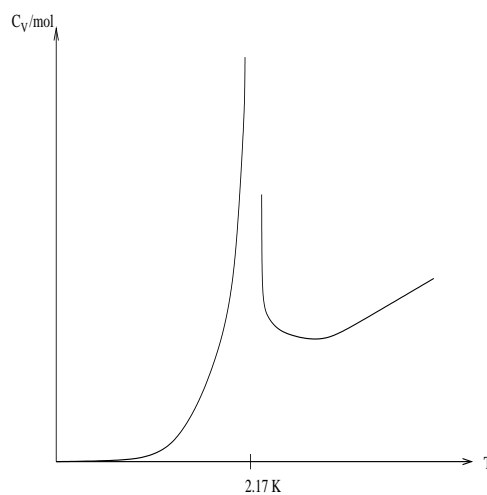
Some basic properties

- Helium 4 atoms have spin zero and therefore satisfy BE stats.
- These atoms have no net charge (inert gas), so the only interatomic forces are the weak van der Waals forces (we will talk about this in greater detail when we deal with interacting systems of particles)
- Thus, it is a liquid right down to the absolute zero of temperature and requires large pressures (~ 25 atm) to solidify it under these conditions.
- These properties can be explained by the fact that the van der Waals forces are too weak to overcome the thermal agitation of the zero-point motion (which is a consequence of the fact that, in the liquid/solid phase, these atoms are highly localized and, due to the Heisenberg uncertainty principle, must therefore have some momentum and kinetic energy.
- At temperatures above 2.17K, ^4He behaves as a normal liquid, called **helium I**, and boils at 4.2K.
- Below 2.17K, ^4He is referred to as **helium II**, and demonstrates some fascinating properties, such as zero entropy and no viscosity.

- The explanation for these properties is that, at these temperatures, helium II behaves as a mixture of two fluids: a normal fluid and a *superfluid*. Below is shown a PT phase diagram for helium 4.



- It would appear that this phenomenon is related to BE condensation, where there we were dealing with two types of particles: those that compose the condensate N_1 and those in excited states $N_{\epsilon>0}$, and here we have two types of fluids in a *liquid* phase. It is because of these differences (i.e. on the one-hand BEC deals with a non-interacting indistinguishable gas, whereas for a liquid interactions between atoms play an important role), however, that discrepancies arise between the predictions of BE theory and experiment.
- The transition from Helium I to Helium II has its signature in the anomalous behaviour of the heat capacity, which has a shape similar to the Greek letter lambda and, hence, this transition is referred to as the *lambda point*.



- This point was measured by F. London to be at $T_c = 3.13K$, where as the experimental lambda point is at $T_\lambda = 3.13K$
- As corroborating evidence for the existence of BEC, we note that ^3He has *no* superfluid transition near 2 K. However, below 3 millikelvin, ^3He has two superfluid phases. This is due to the interaction of pairs of ^3He atoms which are brought together due to the interaction of their nuclear magnetic moments with the surrounding atoms. This phenomenon is analogous to superconductivity (whereby pairs of electrons are held together via interactions with the vibrating lattice of ions). At low T , these pairs condense into the superconducting state, forming a superfluid Bose-Einstein condensate.

