Lecture 10
Planck Distribution

We will now consider some nice applications using our canonical picture. Specifically, we will derive the so-called Planck Distribution and demonstrate that it describes two completely different phenomena: (1) Black-body radiation (BBR); and (2) Einstein model of a solid.

10.1 Description of Einstein model and Black-Body Radiation

1. Black-Body Radiation (BBR) A “black body” absorbs all radiation falling on it. From a statistical point of view, we will derive an expression for the spectrum of electromagnetic radiation in thermal equilibrium within a cavity. Our sun is a good example of a black body and we will consider the emission spectrum for such a case.

- Let’s begin by considering electromagnetic waves within a box (just like a string inside a box attached to two ends).
- These waves can be in different modes, where each mode corresponds to a characteristic frequency, \( \omega = 2\pi f \)

\[
\begin{align*}
\omega_a & \\
\omega_b &
\end{align*}
\]

Figure 10.1: Different wave patterns corresponding to different modes for electromagnetic radiation in a cavity

- These modes can be “excited” in units of the quantum of energy \( \hbar \omega \) such that the energy can be written as,

\[
\epsilon_s = (s + 1/2)\hbar \omega \quad (10.1)
\]

where we have explicitly included the zero-point energy (but, as noted before, only energy differences are physically significant so we may just consider the zero-point energy to be zero.)
• The figure above depicts different modes for a *single* photon. If we “stuff” more photons into the box, then there will be a greater contribution to any given mode,

![Figure 10.2: Two photon occupancy for the two different modes, \( \omega_a \) and \( \omega_b \)](image)

2. **Einstein Model of a Solid**

• Recall our simple model for an Einstein solid which is composed of atoms which can vibrate about their equilibrium or “frozen” positions.

  **Key Assumptions:**
  (a) each atom can vibrate independently and,
  (b) the atoms vibrate with simple harmonic motion.

  This crude model does not account for coupling between the atoms which we will later address with the **Debye Model**

3. **Differences between the Einstein model and BBR**

• The harmonic oscillators treated in the Einstein model are **localized** whereas the E&M waves of our cavity mode are **distributed** throughout the interior of the box.

• For the oscillator, the label \( s \) in the energy \( \epsilon_s = s\hbar\omega \) denotes the quantum number, whereas for a given E&M mode \( s \) gives the number of photons in the mode.

• Nonetheless, the statistical physics of the two problems is similar, since the photons do not interact with each other.

### 10.2 Planck Distribution

1. What is our partition function (for either model)?

\[
Z = \sum_{s=0}^{\infty} e^{-s\beta\hbar\omega} .
\]  

(10.2)

If we let \( x = \exp(-\beta\hbar\omega) \), then we note that since \( x < 1 \) the above infinite series converges and when summed (show in assignment) gives us,

\[
Z = \frac{1}{1 - \exp(-\beta\hbar\omega)} ,
\]

(10.3)

which, you will note is *independent* of \( s \).
2. What about probabilities? The probability that the system is in the state \( s \) with energy \( \epsilon_s = \hbar \omega \) is simply,

\[
P(s) = \frac{e^{-s\beta \hbar \omega}}{Z}
\]

(10.4)

3. How do we calculate the thermal average of \( s \)?

\[
\langle s \rangle = \sum_{s=0}^{\infty} sP(s)
\]

(10.5)

If we let \( y = \beta \hbar \omega \) then we may write,

\[
\sum_{s=0}^{\infty} se^{-sy} = -\frac{d}{dy} \sum_{s=0}^{\infty} e^{-sy} = -\frac{dZ}{dy}
\]

\[
= -\frac{d}{dy} \left( \frac{1}{1 - \exp(-y)} \right)
\]

\[
= \frac{e^{-y}}{(1 - e^{-y})^2}
\]

Therefore,

\[
\langle s \rangle \sum_{s=0}^{\infty} \frac{se^{-sy}}{Z} = \frac{e^{-y}}{(1 - e^{-y})^2} (1 - e^{-y})
\]

\[
= \frac{e^{-y}}{1 - e^{-y}} = \frac{1}{e^y - 1}
\]

Thus, we arrive at the Planck Distribution function:

\[
\langle s \rangle = \frac{1}{e^{\beta \hbar \omega} - 1}
\]

(10.6)

This equation describes the thermal average number of photons in a single mode of frequency \( \omega \). Alternatively, it is the average number of phonons (lattice vibrations) for a given mode in an elastic solid.

4. The average energy for a given mode \( \omega \) is then given as,

\[
\langle \epsilon \rangle = \langle s \rangle \hbar \omega = \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}
\]

(10.7)

5. Although we said that since we are considering energy differences we need not include the zero point energy \( \epsilon_0 = \hbar \omega / 2 \), let us include this now for completeness and calculate
the partition function,

\[ Z = \sum_{s=0}^{\infty} e^{-\beta \hbar \omega (1/2 + s)} \]

\[ = e^{-\beta \hbar /2} \sum_{s=0}^{\infty} e^{-s \beta \hbar \omega} \]

\[ = \frac{e^{-\beta \hbar /2}}{1 - e^{-\beta \hbar \omega}} \]

\[ = \frac{1}{2} \text{csch}(y/2) \quad (10.8) \]

• With the probability given as,

\[ P(s) = \frac{e^{-\beta \hbar \omega (s+1/2)}}{Z} \quad (10.9) \]

we may calculate the average number of photons (or phonons) by first looking at,

\[ \sum_{s=0}^{\infty} (s + 1/2) P(s) = -\frac{d}{dy} \sum_{s} e^{-y(s+1/2)} \]

\[ = -\frac{d}{dy} \left[ e^{-y/2} \left( \frac{1}{1 - e^{-y}} \right) \right] \]

\[ = -\frac{d}{dy} \left( \frac{1}{e^{y/2} - e^{-y/2}} \right) \]

\[ = \frac{1}{2} \frac{d\text{csch}(y/2)}{dy} \]

\[ = \frac{1}{4} \text{csch}(y/2) \text{coth}(y/2) \quad (10.11) \]

Thus, the average is given as,

\[ \langle s + \frac{1}{2} \rangle = \langle s \rangle + \frac{1}{2} = \frac{1}{4} \frac{\text{csch}(y/2) \text{coth}(y/2)}{\frac{1}{2} \text{csch}(y/2)} = \frac{1}{2} \text{coth}(y/2) \quad (10.12) \]

The following plot of the Planck Distribution is shown as a function of the reduced temperature \( T^* \) defined as,

\[ T^* = \frac{k_B T}{\epsilon} \quad (10.13) \]

• Consider the average photon energy for a given mode \( \omega \),

\[ \langle \epsilon \rangle = \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \]

\[ \approx \frac{\hbar \omega}{1 + \beta \hbar \omega - 1} = \frac{\hbar \omega}{\beta \hbar \omega} \approx k_B T \]
Figure 10.3: This figure depicts the average number of photons (or phonons) as a function of the reduced temperature, $T^* = k_B T / \epsilon$. The average number of photons including the zero point has also been included.

### 10.3 Consequences of Planck Distribution: Thermo Quantities

Recall from last lecture that we demonstrated for an ideal system of non-interacting particles the partition function factorizes,

$$Z = z^N$$

Now, if we consider our system of $N$ harmonic oscillators which are all independent and free to oscillate in any of the three directions, the total partition function will factorize further as,

$$Z = z^{3N}$$

1. **Average Energy** Thus, we may calculate the total average energy $\bar{E}$ as,

$$\bar{E} = 3N\hbar\omega \left[ \frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right]$$  \hspace{1cm} (10.14)

where we have again included the zero-point energy.

2. **Heat Capacity**

$$C_v = \left( \frac{\partial E}{\partial T} \right)_v = 3N k_B \frac{y^2 e^y}{(e^y - 1)^2}$$  \hspace{1cm} (10.15)

where again we are dealing with $3N$ degrees of freedom.

- Let us write, $y = \Theta_E / T$ where,

$$\Theta_E \equiv \frac{\hbar \omega}{k_B}$$  \hspace{1cm} (10.16)

is defined as the **Einstein Temperature**. Now let us consider different temperature limits.
• **High Temperature Limit: \( T \ll \Theta_E, y \gg 1 \)**

\[
C_v = 3Nk_B y^2 \frac{(1 + y + \cdots)}{(y^2/2 + \cdots)^2} = 3Nk_B \left(1 + y + \cdots \right) \frac{1}{1 + y/2 + \cdots}^2 = 3Nk_B + \mathcal{O}(y^2)
\]

(10.17)

Therefore, \( C_v \to 3Nk_B \) in the high temperature limit which is in agreement with the "Equipartition Theorem".

• **Low Temperature Limit, \( T \to 0, y \to \infty \)**

\[
C_v = 3Nk_B y^2 \frac{e^{-y}}{(1 - e^{-y})^2} \approx 3Nk_B \left(\frac{\Theta_E}{T}\right)^2 e^{-\Theta_E/T} \to 0
\]

(10.18)

which is what it should be to satisfy the 3rd law of thermodynamics. Picture: Note

![Figure 10.4: Behaviour of the Heat Capacity for an Einstein Solid at different temperature extremes. The low temperature behaviour does not match well with experiment (which demonstrates a \( T^3 \) dependence at low \( T \)](image-url)

that the low temperature behaviour is dominated by the exponential in Eq. 10.18. However, experimentally one finds that as \( T \to 0 \),

\[
C_v \approx T^3
\]

(10.19)

This discrepancy can be explained by the fact that the Einstein model assumes that each atom vibrates independently. As noted above, we shall consider a slightly better model, the Debye model, which corrects for this inadequacy in a slightly ad hoc fashion.